

ON DRIVING FUNCTIONS GENERATING QUASISLITS IN THE CHORDAL LOEWNER-KUFAREV EQUATION

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ABSTRACT. We prove that for every $C > 0$ there exists a driving function $U : [0, 1] \rightarrow \mathbb{R}$ such that the corresponding chordal Loewner-Kufarev equation generates a quasislit and $\limsup_{h \downarrow 0} \frac{|U(1) - U(1-h)|}{\sqrt{h}} = C$.

1. INTRODUCTION AND RESULT

Denote by $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ the upper half-plane. A bounded subset $A \subset \mathbb{H}$ is called a (*compact*) *hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. By g_A we denote the unique conformal mapping from $\mathbb{H} \setminus A$ onto \mathbb{H} with *hydrodynamic normalization*, i.e.

$$g_A(z) = z + \frac{b}{z} + \mathcal{O}(|z|^{-2}) \quad \text{for } |z| \rightarrow \infty$$

and for some $b \geq 0$. The quantity $\operatorname{hcap}(A) := b$ is called *half-plane capacity* of A .

The chordal (one-slit) Loewner-Kufarev equation for \mathbb{H} is given by

$$(1.1) \quad \dot{g}_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z \in \mathbb{H},$$

where $U : [0, T] \rightarrow \mathbb{R}$ is a continuous function, the so called *driving function*. For $z \in \mathbb{H}$, let T_z be the supremum of all t such that the solution exists up to time t and $g_t(z) \in \mathbb{H}$. Let $K_t := \{z \in \mathbb{H} \mid T_z \leq t\}$, then $\{K_t\}_{t \in [0, T]}$ is a family of increasing hulls and g_t is the unique conformal mapping of $\mathbb{H} \setminus K_t$ onto \mathbb{H} with hydrodynamic normalization and $g_t(z) = z + \frac{2t}{z} + \mathcal{O}(|z|^{-2})$ for $z \rightarrow \infty$.

If $\gamma : [0, T] \rightarrow \overline{\mathbb{H}}$ is a simple curve, i.e. a continuous, one-to-one function with $\gamma(0) \in \mathbb{R}$ and $\gamma((0, 1]) \subset \mathbb{H}$, then we call the hull $\Gamma := \gamma((0, 1])$ a *slit*. The important connection between slits and equation (1.1) is given by the following Theorem.

Theorem A (Kufarev, Sobolev, Sporysheva). *For any slit Γ with $\operatorname{hcap}(\Gamma) = 2T$ there exists a unique continuous driving function $U : [0, T] \rightarrow \mathbb{R}$ such that the solution g_t of (1.1) satisfies $g_T = g_\Gamma$.*

Proof. The first proof was given by Kufarev et al. in [1]. For a English reference, see [2], p. 92f. \square

Conversely, equation (1.1) does not necessarily generate a slit for a given driving function, see Example 2. A sufficient condition for getting slits was found by J. Lind, D. Marshall and S. Rohde:

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According to [7] and [5] we define a *quasislit* to be the image of $[0, i]$ under a quasiconformal mapping $Q : \mathbb{H} \rightarrow \mathbb{H}$ with $Q(\mathbb{H}) = \mathbb{H}$ and $Q(\infty) = \infty$. In other words, a quasislit is a slit that is a quasia arc approaching \mathbb{R} nontangentially (see Lemma 2.3 in [7]).

Let $\text{Lip}(\frac{1}{2})$ be the set of all continuous functions $U : [0, T] \rightarrow \mathbb{R}$ with

$$|U(t) - U(s)| \leq c\sqrt{|s - t|} \quad \text{for some } c > 0 \quad \text{and all } s, t \in [0, T].$$

Now the following connection between $\text{Lip}(\frac{1}{2})$ and quasislits holds.

Theorem B. (Theorem 1.1 in [7] and Theorem 2 in [5]) *If Γ is a quasislit with driving function U , then $U \in \text{Lip}(\frac{1}{2})$. Conversely, if $U \in \text{Lip}(\frac{1}{2})$ and for every $t \in [0, T]$ there exists an $\varepsilon > 0$ such that*

$$\sup_{\substack{r, s \in [0, T] \\ |r - t|, |s - t| < \varepsilon}} \frac{|U(r) - U(s)|}{\sqrt{|r - s|}} < 4,$$

then Γ is a quasislit.

The Hölder constant 4 in Theorem B is not necessary for generating quasislits: For any $s \in [0, T)$, the “right pointwise Hölder norm”, i.e. the value

$$\limsup_{h \downarrow 0} \frac{|U(s + h) - U(s)|}{\sqrt{h}}$$

can get arbitrarily large, as the driving function $U(t) = c\sqrt{t}$ shows:

Example 1. *Let $U(t) = c\sqrt{t}$ for an arbitrary $c \in \mathbb{R}$. In this case, the one-slit equation (1.1) can be solved explicitly and one obtains for the generated hull K_t at time $t : K_t = \gamma([0, t])$ with $\gamma(t) = 2\sqrt{t} \left(\frac{\pi}{\phi} - 1 \right)^{\frac{1}{2} - \frac{\phi}{\pi}} e^{i\phi}$, i.e. K_t is a line segment with angle ϕ , see Example 4.12 in [2]. The connection between c and ϕ is given by*

$$c(\phi) = \frac{2(\pi - 2\phi)}{\sqrt{\phi(\pi - \phi)}}, \quad \phi(c) = \frac{\pi}{2} \left(1 - \frac{c}{\sqrt{c^2 + 16}} \right).$$

The aim of this paper is to show that the “left pointwise Hölder-norm” can also get arbitrarily large within the space of all driving functions that generate quasislits.

Theorem 1. *For every $C > 0$, there exists a driving function $U : [0, 1] \rightarrow \mathbb{R}$ that generates a quasislit and satisfies*

$$\limsup_{h \downarrow 0} \frac{|U(1) - U(1 - h)|}{\sqrt{h}} = C.$$

We will prove Theorem 1 in Section 3. In Section 2 we discuss some necessary and sufficient conditions on driving functions for generating slits.

2. CONFORMAL WELDING IN THE LOEWNER EQUATION

In the following we denote by $B(z, r)$, $z \in \mathbb{C}$, $r > 0$, the Euclidean ball with centre z and radius r , i.e. $B(z, r) = \{w \in \mathbb{C} \mid |z - w| < r\}$.

As already mentioned, there are continuous driving functions that generate hulls which are not slits.

Example 2. *Consider the driving function $U(t) = c\sqrt{1 - t}$ with $c \geq 4$ and $t \in [0, 1]$: The generated hull K_t is a slit for every $t \in (0, 1)$, but at $t = 1$ this slit hits the real axis at an angle φ which can be calculated directly (see [4], chapter 3):*

$$\varphi = \pi - \frac{2\pi\sqrt{c^2 - 16}}{\sqrt{c^2 - 16} + c}.$$

Its complement with respect to \mathbb{H} has two connected components and K_1 is the closure of the bounded component and consequently not a slit, see Figure 1.

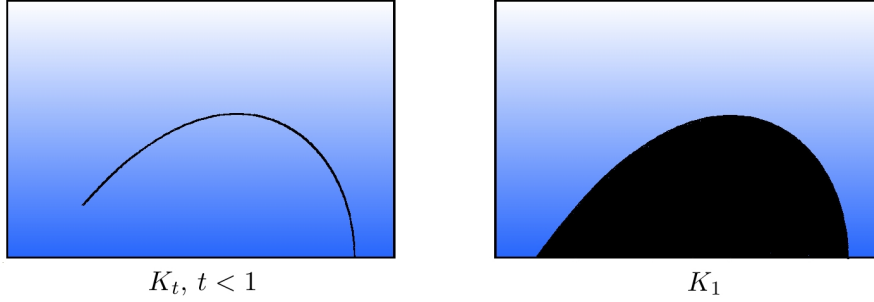


FIGURE 1. Example 2 with $c = 5$.

There are further, more subtle obstacles preventing the one-slit equation from producing slits as the following example shows.

Example 3. *There exists a driving function $U \in \text{Lip}(\frac{1}{2})$ such that K_t is a simple curve γ for all $t \in [0, T)$ and for $t \rightarrow T$ this curve wraps infinitely often around $B(2i, 1)$. Hence $K_T = \gamma[0, 1) \cup \overline{B(2i, 1)}$ is not locally connected, see Example 4.28 in [2].*

In order to distinguish between these two kinds of obstacles, one has introduced two further notions, which are more general than “the hull is a slit”.

For this we have to take a look at the so called backward equation. Let $U : [0, T] \rightarrow \mathbb{R}$ be continuous. Furthermore, let g_t be the solution to (1.1). The backward equation is given by

$$(2.1) \quad \dot{x}(t) = \frac{-2}{x(t) - U(T-t)}, \quad x(0) = x_0 \in \overline{\mathbb{H}} \setminus \{U(T)\}.$$

For $x_0 \in \mathbb{H}$, the solution $x(t)$ exists for all $t \in [0, T]$ and the function $f_T : x_0 \mapsto x(T)$ satisfies $f_T = g_T^{-1}$.

For $x_0 \in \mathbb{R} \setminus \{U(T)\}$, the solution may not exist for all $t \in [0, T]$. If a solution ceases to exist, say at $t = s$, it will hit the singularity, i.e. $\lim_{t \rightarrow s} x(t) = U(T-s)$.

Now suppose that two different solutions $x(t), y(t)$ with $x(0) = x_0 < y_0 = y(0)$ meet the singularity at $t = s$, i.e. $\lim_{t \rightarrow s} x(t) = \lim_{t \rightarrow s} y(t) = U(T-s)$. Then x_0 and y_0 lie on different sides with respect to $U(T)$, that is $x_0 < U(T) < y_0$. Otherwise the difference $y(t) - x(t)$ would satisfy

$$\dot{y}(t) - \dot{x}(t) = \frac{2(T-t)(y(t) - x(t))}{(y(t) - U(T-t))(x(t) - U(T-t))} > 0$$

for all $0 \leq t < s$ and thus, $\lim_{t \rightarrow s} (x(t) - y(t)) = 0$ would be impossible.

Consequently, for any $s \in (0, T]$, there are at most two initial values so that the corresponding solutions will meet in $U(T-s)$.

Definition 2. Let $\{K_t\}_{t \in [0, T]}$ be a family of hulls generated by the one-slit equation with driving function U .

- (1) $\{K_t\}_{t \in [0, T]}$ is *welded* if for every $s \in (0, T]$ there exist exactly two real values x_0, y_0 with $x_0 < U(T) < y_0$ such that the corresponding solutions $x(t)$ and $y(t)$ of (2.1) with $x(0) = x_0, y(0) = y_0$ satisfy $x(s) = y(s) = U(T-s)$.
- (2) $\{K_t\}_{t \in [0, T]}$ is *generated by a curve* if there exists a simple curve $\gamma : [0, T] \rightarrow \overline{\mathbb{H}}$, such that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ for every $t \in [0, T]$.

Remark 3. Several properties of hulls that are generated by curves are described in [2], Section 4.4. The notion of welded hulls was introduced in [7] for the radial Loewner equation. The chordal case is considered in [5]. Informally speaking, welded hulls have a left and a right side.

If $\{K_t\}_{t \in [0, T]}$ is welded and the interval $I = [a, b]$ is the cluster set of K_T with respect to g_T , then $a < U(T) < b$ and for every $a \leq x_0 < U(T)$ there exists $U(T) < y_0 \leq b$ such that the solutions to (2.1) with initial values x_0 and y_0 hit the singularity at the same time. This gives a welding homeomorphism $h : [a, b] \rightarrow [a, b]$ by defining $h(x_0) := y_0$, $h(y_0) := x_0$, $h(U(T)) := U(T)$.

Furthermore, the hulls $\{K_t\}_{t \in [0, T]}$ describe a quasislit if and only if they describe a slit and the homeomorphism h is a quasymmetric function, see Lemma 6 in [5].

The hull of Example 3, which is sketched in picture a) of Figure 2, is not generated by a curve. Picture c) shows an example of a hull that is generated by a curve. Here, the curve hits itself and the real axis and consequently, this hull is not welded. The hulls in picture b), which form a “topologist’s sine curve” approaching a compact interval on \mathbb{R} , are simple curves before the “sine curve” touches \mathbb{R} , but then, the corresponding hull is neither welded nor generated by a curve.

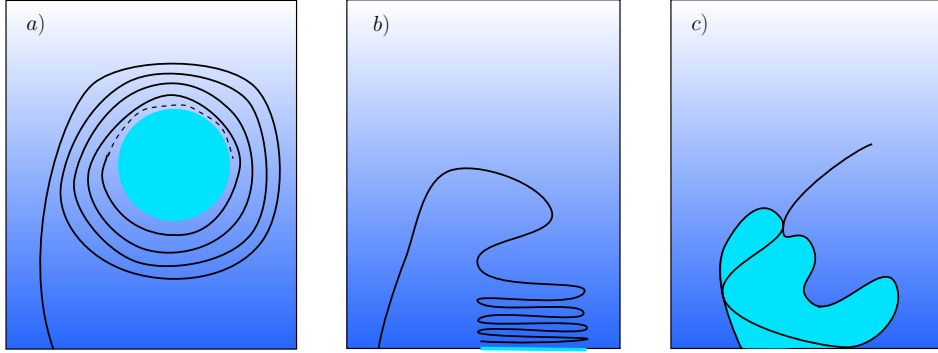


FIGURE 2. Three cases of hulls that are not slits.

Proposition 4. The hulls $\{K_t\}_{t \in [0, T]}$ describe a slit if and only if $\{K_t\}_{t \in [0, T]}$ is generated by a curve and it is welded.

Proof. For the non-trivial direction of the statement, see Lemma 4.34 in [2]. \square

The following statement follows directly from the proof of Lemma 3 in [5]. For the sake of completeness we include the proof.

Proposition 5. Let $\{K_t\}_{t \in [0, T]}$ be family of hulls generated by the one-slit equation. The following statements are equivalent:

- a) $\{K_t\}_{t \in [0, T]}$ is welded.
- b) For every $\tau \in [0, T)$ there exists $\varepsilon > 0$ such that for all $x_0 \in \mathbb{R} \setminus \{U(\tau)\}$ the solution $x(t)$ of

$$(2.2) \quad \dot{x}(t) = \frac{2}{x(t) - U(t)}, \quad x(\tau) = x_0,$$

does not hit $U(t)$ for $t < T$ and satisfies $|x(T) - U(T)| > \varepsilon$.

Proof. a) \implies b) : Firstly, the solution $x(t)$ to (2.2) exists locally, say in the interval $[\tau, T^*)$, $T^* \leq T$. Now we know that there are x_1^0, x_2^0 with $x_1^0 < x_2^0$, such that the solutions $x_1(t)$ and $x_2(t)$ to equation (2.1) with initial values x_1^0 and x_2^0 respectively hit the singularity

$U(\tau)$ at $s = T - \tau$. But this implies that $x(t)$ can be extended to the interval $[0, T^*]$ with $|U(T^*) - x(T^*)| < \varepsilon$, where $\varepsilon := \min\{U(T^*) - x_1(T - T^*), x_2(T - T^*) - U(T^*)\}$.

$b) \implies a)$: Let $\tau = T - s$. We set $a_n := U(\tau) - \frac{1}{n}$ for all $n \in \mathbb{N}$. The solution $x_n(t)$ of (2.2) with initial value a_n exists up to time T . Hence we can define $\xi_n := x_n(T)$ for all $n \geq N$ and we have $U(T) - \xi_n > \varepsilon$. The sequence ξ_n is increasing and bounded above, and so it has a limit $x_0 < U(T)$. Then the solution $x(t)$ of (2.1) with x_0 as initial value satisfies $\lim_{t \rightarrow s} x(t) = \lim_{n \rightarrow \infty} a_n = U(\tau) = U(T - s)$.

The second value y_0 can be obtained in the same way by considering the sequence $U(\tau) + \frac{1}{n}$ instead of a_n . \square

Remark 6. *The proof of Proposition 3.1 in [9] implies the following necessary condition for getting welded hulls: If K_T is welded, then, for every $s \in (0, T]$, we have*

$$\begin{aligned} \limsup_{h \downarrow 0} \frac{|U(s) - U(s-h)|}{\sqrt{h}} &< 4 \quad (\text{“regular case”}), \text{ or} \\ \liminf_{h \downarrow 0} \frac{|U(s) - U(s-h)|}{\sqrt{h}} &< 4 \leq \limsup_{h \downarrow 0} \frac{|U(s) - U(s-h)|}{\sqrt{h}} \quad (\text{“irregular case”}). \end{aligned}$$

Conversely, if the regular condition holds for every $s \in (0, T]$, then K_T is welded, see [5], Section 5.

For driving functions which are “irregular” in at least one point, it is somehow harder to find out whether the generated hulls are welded or not. Here we derive a sufficient condition for a very special case. This case will be needed in the proof of Theorem 1.

Lemma 7. *Let $U : [0, 1] \rightarrow \mathbb{R}$ be continuous with $U(1) = 0$ and let $\{K_t\}_{t \in [0, 1]}$ be the hulls generated by equation (1.1). Suppose that there are two increasing sequences s_n, t_n of positive numbers with $s_n, t_n \rightarrow 1$, such that for*

$$\overline{M}_n := \max_{s_n \leq t \leq 1} \{U(t)\} \quad \text{and} \quad \underline{M}_n := \min_{t_n \leq t \leq 1} \{U(t)\}$$

the two inequalities

$$4(1 - s_n) + U(s_n)^2 - 2U(s_n)\overline{M}_n > 0 \quad \text{and} \quad 4(1 - t_n) + U(t_n)^2 - 2U(t_n)\underline{M}_n > 0$$

hold for all $n \in \mathbb{N}$. If K_t is welded for all $t \in (0, 1)$, then so is K_1 .

Proof. Let $\tau \in [0, 1)$ and $x_0 \in \mathbb{R} \setminus \{U(\tau)\}$. By Proposition 5 we know that the solution $x(t)$ of the initial value problem

$$x(\tau) = x_0, \quad \dot{x}(t) = \frac{2}{x(t) - U(t)}$$

exists until $t = 1$ and we have to show that there is a lower bound for $|x(1) - U(1)|$ which is independent of x_0 .

Assume that $x_0 < U(\tau)$. Then $x(t)$ is decreasing and we have $x(s_m) < U(s_m)$ with $m := \min\{n \in \mathbb{N} \mid s_n \geq \tau\}$. The initial value problem

$$\dot{y}(t) = \frac{2}{y(t) - \overline{M}_m}, \quad y(s_m) = x(s_m),$$

has the solution $y(t) = \overline{M}_m - \sqrt{(M_m - x(s_m))^2 + 4(t - s_m)}$. Now we have

$$\dot{x}(t) \leq \frac{2}{x(t) - \overline{M}_m} \quad \text{for all } t \in [s_m, 1)$$

and $x(s_m) = y(s_m)$. Consequently,

$$\begin{aligned} x(1) &\leq y(1) = \overline{M}_m - \sqrt{(\overline{M}_m - x(s_m))^2 + 4(1 - s_m)} \\ &< \underbrace{\overline{M}_m - \sqrt{(\overline{M}_m - U(s_m))^2 + 4(1 - s_m)}}_{=: L_1} < \overline{M}_m - \sqrt{\overline{M}_m^2} = 0. \end{aligned}$$

The case $x_0 > U(\tau)$ can be treated in the same way and gives a bound $L_2 > 0$ for $x(1) = x(1) - U(1)$. Thus, the condition in Proposition 5 b) is satisfied for $\varepsilon = \min\{L_1, L_2\}$ and it follows that K_1 is welded. \square

Corollary 8. *If K_t is welded for all $t \in (0, 1)$ and there are two increasing sequences s_n, t_n of positive numbers with $s_n, t_n \rightarrow 1$, and $U(s_n) \leq U(1) \leq U(t_n)$ for all n , then K_1 is welded, too.*

Proof. Without loss of generality we can assume $U(1) = 0$. We can apply Lemma 7 as

$$\begin{aligned} 4(1 - s_n) + U(s_n)^2 - 2U(s_n)\overline{M}_n &> -2U(s_n)\overline{M}_n \geq 0 \quad \text{and} \\ 4(1 - t_n) + U(t_n)^2 - 2U(t_n)\underline{M}_n &> -2U(t_n)\underline{M}_n \geq 0. \end{aligned}$$

\square

3. PROOF OF THEOREM 1

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $C > 0$. First we construct the driving function U , which is shown in Figure 3 for the case $C = 5$.

We set $U(r_n) := 0$ with $r_n := 1 - \frac{1}{2^n}$ for all $n \geq 0$. The mean value of r_n and r_{n+1} is equal to $w_n := 1 - \frac{3}{2^{n+2}}$ and here we define

$$U(w_n) := C\sqrt{\frac{3}{2^{n+2}}} \quad \text{for } n \geq 0.$$

Now we define $U(t)$ for $t \in [0, 1)$ by linear interpolation, so that

$$\begin{aligned} U(t) &= C\sqrt{3 \cdot 2^{n+2}} \cdot (t - r_n) \quad \text{for } t \in [r_n, w_n] \quad \text{and} \\ U(t) &= C\sqrt{3 \cdot 2^{n+2}} \cdot (r_{n+1} - t) \quad \text{for } t \in [w_n, r_{n+1}]. \end{aligned}$$

By defining $U(1) := 0$ we now have a continuous driving function and

$$\limsup_{h \downarrow 0} \frac{|U(1) - U(1 - h)|}{\sqrt{h}} = \lim_{n \rightarrow \infty} \frac{|U(1) - U(w_n)|}{\sqrt{1 - w_n}} = \lim_{n \rightarrow \infty} \frac{C\sqrt{3/2^{n+2}}}{\sqrt{3/2^{n+2}}} = C.$$

At each $0 \leq t < 1$, the hull K_t produced by this function will be a quasislit according to Theorem B. Thus, we have to show that also K_1 is a slit and that this slit is a quasiaarc.

First, we know that $\{K_t\}_{t \in [0, 1]}$ is welded: This follows directly from Corollary 8 by setting $s_n := t_n := r_n$.

If we scale our hull by $\frac{1}{\sqrt{2}}$, we end up with the new driving function $\tilde{U} : [0, 1/2] \rightarrow \mathbb{R}$, $\tilde{U}(t) = \frac{1}{\sqrt{2}}U(2t)$. However, this is again $U(t)$, confined to the interval $[1/2, 1]$, i.e. $\tilde{U}(t) = U(t + 1/2)$. Geometrically, this means that $g_{1/2}(\overline{K_1 \setminus K_{1/2}})$ is just the same as $\frac{1}{\sqrt{2}}K_1$, the original hull scaled by $\frac{1}{\sqrt{2}}$.

If $f := g_{1/2}^{-1}$, and $S_n := \overline{K_{1-1/2^n} \setminus K_{1-1/2^{n-1}}}$, $n \geq 1$, then we have

$$S_{n+1} = f\left(\frac{1}{\sqrt{2}}S_n\right).$$

As the function $z \mapsto f(\frac{1}{\sqrt{2}}z) =: I(z)$ is not an automorphism of \mathbb{H} , the Denjoy–Wolff Theorem implies that the iterates $I^n = (I \circ \dots \circ I)$ converge uniformly on S_1 to a point $S_\infty \in \overline{\mathbb{H}} \cup \{\infty\}$. $S_\infty = \infty$ is not possible as the hull K_1 is a compact set and the case $S_\infty \in \mathbb{R}$ would imply that K_1 is not welded. Consequently $S_\infty \in \mathbb{H}$ and $K_1 = \bigcup_{n \geq 1} S_n \cup \{S_\infty\}$ is a simple curve whose tip is S_∞ .

Now we show that this curve is a quasiarc.

For this, we will use the metric characterization of quasiarcs by Ahlfors’ three point property (also called *bounded turning property*, see [3], Section 8.9, or [8], Theorem 1), which says that K_1 is a quasiarc if and only if

$$\sup_{\substack{x, y \in K_1 \\ x \neq y}} \frac{\text{diam}(x, y)}{|x - y|} < \infty,$$

where we denote by $\text{diam}(x, y)$ the diameter of the subcurve of K_1 joining x and y . For $m \in \mathbb{N} \cup \{0\}$ we define $F_m := \bigcup_{k \geq m} S_k \cup \{S_\infty\}$. As K_t is a quasislit for every $t \in (0, 1)$, it suffices to show that

$$(3.1) \quad \sup_{\substack{x, y \in F_m \\ x \neq y}} \frac{\text{diam}(x, y)}{|x - y|} < \infty \quad \text{for one } m \in \mathbb{N}.$$

The set S_n contracts to S_∞ when $n \rightarrow \infty$, in particular $\text{diam}(S_n) \rightarrow 0$.

As I is conformal in $B(S_\infty, \varepsilon)$ for $\varepsilon > 0$ small enough, there is an $N = N(\varepsilon) \in \mathbb{N}$, such that $S_n \subset B(S_\infty, \varepsilon)$ for all $n \geq N$.

S_∞ is a fixpoint of $I(z)$ and so $|I'(S_\infty)| < 1$. Otherwise, I would be an automorphism of \mathbb{H} .

Now, for $x \in S_{N+n}$, $n \geq 0$, we have $I(x) \in S_{N+n+1}$ and

$$\begin{aligned} |I(x) - S_\infty| &= |I(x) - I(S_\infty)| = |I'(S_\infty)(x - S_\infty) + \mathcal{O}(|x - S_\infty|^2)| \\ &= |I'(S_\infty) + \mathcal{O}(|x - S_\infty|)| \cdot |x - S_\infty| = |I'(S_\infty)| |1 + \mathcal{O}(|\varepsilon|)| \cdot |x - S_\infty|. \end{aligned}$$

Consequently we can pass on to a smaller ε (and larger N) such that $\text{dist}(S_\infty, S_{N+n+1}) \leq c \text{dist}(S_\infty, S_{N+n})$ with $c < 1$ and for all $n \geq 0$. Hence $S_{N+n} \subset B(S_\infty, c^n \varepsilon)$.

Furthermore, for $x, y \in F_{N+n}$ we have

$$|I(x) - I(y)| = |I'(x) + \mathcal{O}(c^n \varepsilon)| \cdot |x - y| = |I'(S_\infty) + \mathcal{O}(c^n \varepsilon)| \cdot |x - y|.$$

Hence there exist positive constants a_1, a_2 with $1 - a_2 c^n > 0$ such that

$$(3.2) \quad (1 - a_2 c^n) |I'(S_\infty)| |x - y| \leq |I(x) - I(y)| \leq (1 + a_1 c^n) |I'(S_\infty)| |x - y|.$$

Thus

$$(3.3) \quad \text{diam}(I(x), I(y)) \leq (1 + a_1 c^n) |I'(S_\infty)| \text{diam}(x, y).$$

Now we will show (3.1) for $m = N$. Let $x, y \in F_N$ with $x \neq y$. We assume that $\text{diam}(x, S_\infty) \geq \text{diam}(y, S_\infty)$. In particular, $x \neq S_\infty$ and thus there is a $k \geq 0$ and an $\hat{x} \in S_N$ such that $x = I^k(\hat{x})$. Let $\hat{y} \in F_N$ be defined by $y = I^k(\hat{y})$. First note that

$$\sup_{\substack{a \in S_N, b \in F_N \\ a \neq b}} \frac{\text{diam}(a, b)}{|a - b|} =: C < \infty,$$

for K_t is a quasislit for every $t \in (0, 1)$. Now we get with (3.2) and (3.3):

$$\begin{aligned}
\frac{\text{diam}(x, y)}{|x - y|} &= \frac{\text{diam}(I^k(\hat{x}), I^k(\hat{y}))}{|I^k(\hat{x}) - I^k(\hat{y})|} \leq \frac{(1 + a_1 c^{k-1})|I'(S_\infty)| \text{diam}(I^{k-1}(\hat{x}), I^{k-1}(\hat{y}))}{(1 - a_2 c^{k-1})|I'(S_\infty)||I^{k-1}(\hat{x}) - I^{k-1}(\hat{y})|} = \\
&= \frac{(1 + a_1 c^{k-1})}{(1 - a_2 c^{k-1})} \cdot \frac{\text{diam}(I^{k-1}(\hat{x}), I^{k-1}(\hat{y}))}{|I^{k-1}(\hat{x}) - I^{k-1}(\hat{y})|} \leq \dots \leq \\
&\leq \prod_{j=0}^{k-1} \frac{(1 + a_1 c^j)}{(1 - a_2 c^j)} \cdot \frac{\text{diam}(\hat{x}, \hat{y})}{|\hat{x} - \hat{y}|} \leq \prod_{j=0}^{k-1} \frac{(1 + a_1 c^j)}{(1 - a_2 c^j)} \cdot C \leq C \prod_{j=0}^{\infty} \frac{1 + a_1 c^j}{1 - a_2 c^j} = \\
&= C \prod_{j=0}^{\infty} (1 + a_1 c^j) / \prod_{j=0}^{\infty} (1 - a_2 c^j) < \infty.
\end{aligned}$$

The two Pochhammer products converge because $|c| < 1$. Consequently, K_1 is a quasislit. \square

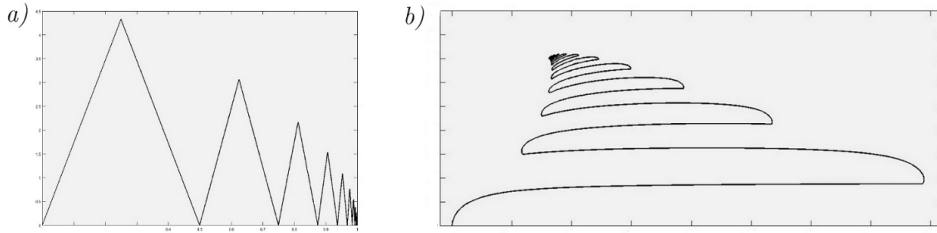


FIGURE 3. The driving function U from Proposition 1 with $C = 5$ (left) and the generated quasislit (right).

Remark 9. *The argument that U from the proof of Proposition 1 generates a slit holds for a more general case:*

Let $U : [0, 1] \rightarrow \mathbb{R}$ be continuous with $U(1) = 0$. Call such a function d -similar with $0 < d < 1$ if $V(t) := U(1 - t)$ satisfies

$$d \cdot V(t/d^2) = V(t) \quad \text{for all } 0 < t \leq d^2.$$

Every d -similar function can be constructed by defining $V(1)$ arbitrarily, putting $V(d^2) = d \cdot V(1)$ and then defining $V(t)$ for $d^2 < t < 1$ such that V is continuous in $[d^2, 1]$. Then, V is uniquely determined for all $0 \leq t \leq 1$. Now we have: If U is d -similar such that it produces a slit for all $0 \leq t < 1$ and the hull at $t = 1$ is welded, then K_1 is a slit, too.

Example 4. *There exists a driving function $V : [0, 1] \rightarrow \mathbb{R}$ that is "irregular" at infinitely many points and generates a quasislit:*

Let U be the driving function from the proof of Proposition 1. We construct $V : [0, 1] \rightarrow \mathbb{R}$ by sticking pieces of U appropriately together. For $n \geq 0$ let

$$V(t) := U((t - (1 - 1/2^n)) \cdot 2^{n+1}) / \sqrt{2^n} \quad \text{for } t \in [1 - 1/2^n, 1 - 1/2^{n+1}],$$

and $V(1) := 0$. Then V is "irregular" at $1 - 1/2^n$ for all $n \geq 1$ and it produces a quasislit: The hull generated at $t = 1/2$ is a quasislit by Theorem 1. Now one can repeat the proof of Theorem 1 to show that the whole hull is a quasislit, too.

These examples together with Theorem B suggest the following question: Does U generate a quasislit if U generates a slit and $U \in \text{Lip}(\frac{1}{2})$?

The answer is no: There are $\text{Lip}(\frac{1}{2})$ -driving functions that generate slits with positive area. These slits cannot be quasislits as they are not uniquely determined by their welding homeomorphisms, see Corollary 1.4 in [6].

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